Consistency in Formulation of Spin Current and Torque Associated with a Variance of Angular Momentum

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(Received 14 August 2005; published 13 February 2006)

Based on the Noether’s theorem, we develop systematically and rigorously the spin-dependent formulation of the conservation laws. The effect of the electronic polarization due to the spin-orbit coupling is included in the Maxwell equations. The polarization is related to the antisymmetric components of spin current, and it provides a possibility to measure the spin current directly. The variances of spin and orbit angular momentum currents imply a torque on the “electric dipole” associated with the moving electron. The dependencies of the torque on the polarization and the force on the motions of spin-polarized electrons in a two-dimensional electron gas with the Rashba spin-orbit coupling are discussed.

DOI: 10.1103/PhysRevLett.96.066601

PACS numbers: 72.10Bg, 71.70.Ej, 72.25.–b

The predictions of the extrinsic and intrinsic spin Hall effects (SHE) due to the spin-orbit (SO) interaction in metals and doped semiconductors [1–5] has attracted considerable interest recently. The spin Hall current is a motion of carriers with opposite spins in opposite directions perpendicular to the driving electric field. A corollary of the spin Hall current is the spin accumulation it induces at the edges in the presence of spin-flip scattering [6]. The spin accumulation at the sample edges has recently been detected by optical methods [7–9]. The effect of impurity scattering on intrinsic SHE has also been investigated [10–12].

Several fundamental questions have yet to be answered before the SHE is fully understood. For example, no consensus has been reached about the definition of the spin current operator in spin-orbit coupled systems. The conventional spin current operator, defined as a symmetric product of the Pauli spin and the velocity operators, \( J^S_\sigma = (\hbar/4) [\sigma, \mathbf{v}] \) [3–5], does not satisfy a continuity equation \([13–15]\), i.e., \( \partial \rho^S_\sigma / \partial t + \nabla \cdot J^S_\sigma \neq 0 \), where \( \rho^S_\sigma = (\hbar/2) \psi^\dagger \sigma \psi\) is the spin density operator. Indeed, the spin current associated with the translational and the rotational degrees of freedom must be defined rigorously and consistently with the definitions of force and torque. Unlike the charge current, a conserved spin current cannot be obtained within the Schrödinger equation in which relativistic corrections are included by a weak SO interaction. Instead, a fully relativistic formulation of the angular momentum conservation laws is required.

In this Letter, we formulate the linear and angular momentum current tensors based on Noether’s theorem. Starting with the Dirac equations of motion and relativistic continuity equations, we recover in the weakly relativistic limit the Maxwell equations in the presence of SO coupling, in which the conservation of various currents is guaranteed. By considering an external electromagnetic field, the force and the torque on the spin carriers can be identified rigorously. The characteristics of the spin current can be obtained by investigating its antisymmetric part that is related to the spin polarization by intrinsic electric dipoles. We predict that it is possible to detect the spin current by the electrical polarization.

The system of Dirac fermions (\( \Psi, \bar{\Psi} \)) coupled to an electromagnetic field (\( A_\mu \)) is described by the relativistic covariant Lagrangian \( \mathcal{L} = -i(\bar{\Psi} / 2) \gamma_\mu \bar{\Psi} \mu - mc^2 \bar{\Psi} \Psi - \bar{\Psi} \gamma_\mu \gamma_5 (1/4) \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \). Here the four-vector notation is used [16] and \( c \)

\[
\gamma_\mu = \begin{pmatrix} 0 & -i\sigma & 0 \\ i\sigma & 0 & -I \\ 0 & I & 0 \end{pmatrix}
\]

with the Pauli matrices \( \sigma \). \( \Pi_\mu = -i\hbar \partial_\mu - (e/c) A_\mu \) is a covariant momentum operator, \( \mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic tensor, and we use the multiplication rule \( f\Pi_\mu g = f \Pi_\mu g - (\Pi_\mu^f g) \). The Dirac and Maxwell equations for the fields \( \Psi, \bar{\Psi} \), and \( \mathcal{F}_{\mu\nu} \) follow from the Euler-Lagrangian equations associated with \( \mathcal{L} \). From the Dirac equation, the Schrödinger-Pauli Hamiltonian with SO coupling can be obtained by the standard nonrelativistic approximation (NRA). The source terms in the Maxwell equations, i.e., the charge and the current densities in the absence of the SO coupling, can be treated with the same NRA procedure. To leading order in \( \nu^2 / c^2 \), the spin-dependent forms can be explicitly found to be

\[
\rho^C = \rho^S_\sigma - \nabla \cdot \mathbf{P} \tag{1}
\]

and

\[
\mathbf{J}^C = \mathbf{J}^S_0 + e \mathbf{v} \times \mathbf{m} + \partial \mathbf{P} / \partial t, \tag{2}
\]

where \( \rho^S_\sigma = \psi^\dagger \psi\) and \( \mathbf{J}^S_0 = \psi^\dagger v \psi\) are the charge and the current densities in the absence of the SO coupling. Here \( \mathbf{v} = (1/m) \mathbf{P} + (\hbar e/4m c^2) \mathbf{E} \times \sigma \) is the velocity operator and \( \mathbf{m} = (e\hbar /2mc) \psi^\dagger \sigma \psi\) the intrinsic magnetic moment.
of the electron. The term \( e \mathbf{V} \times \mathbf{m} \) leads to the “spin magnetization current” [18]. The extra terms \(-\nabla \cdot \mathbf{P}\) and \( \partial \mathbf{P}/\partial t \) in the charge and the current densities are an induced electrical polarization of the electron moving in the pseudomagnetic field due to the SO coupling:

\[
\mathbf{P} = \frac{\hbar^2}{8m^2c^2} \nabla \rho_0^C - \frac{\hbar e}{8mc^2} \psi^\dagger \sigma \times \mathbf{B} \psi.
\]  

(3)

By analogy with the electrodynamics in the macroscopic media, an electric displacement vector \( \mathbf{D} = \mathbf{E} + \mathbf{P} \) and magnetic field strength \( \mathbf{H} = \mathbf{B} - \mathbf{m} \) can be defined. The Maxwell equations in Heaviside-Lorentz units then read

\[
\nabla \cdot \mathbf{D} = \rho_0^C, \quad \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c} \mathbf{J}_0^C, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0.
\]  

(4)

These equations describe all electromagnetic phenomena including the spin degree of freedom of charged carriers with SO interaction. The moving electron generates the electromagnetic fields not only by its bare charge \( \rho_0^C \) and current \( \mathbf{J}_0^C \) densities, but also by its electrical polarization \( \mathbf{P} \) and magnetic moment \( \mathbf{m} \). The Maxwell-like Boltzmann equations for spin have been formulated by Bernevig and Zhang [19].

The above charge and current densities satisfy the non-relativistic continuity equation. The idea of a finite polarization due to a moving electron with the intrinsic magnetic moment was pointed out by Frenkel in 1926 [20], which can be understood by invoking the static energy \( \mathcal{E}_P = -\int d\mathbf{r} \mathbf{E} \cdot \mathbf{P} \). Apart from a surface term, \( \mathbf{E} \cdot \mathbf{P} \) can be written in terms of the SO coupling and the Darwin terms in the Schrödinger-Pauli Hamiltonian, i.e.,

\[
\mathcal{E}_P = \int d\mathbf{r} \psi^\dagger H_P \psi
\]

with

\[
H_P = -\frac{\hbar e}{4mc^2} \sigma \cdot (\mathbf{E} \times \mathbf{P}) - \frac{\hbar^2e}{8mc^2} \nabla \cdot \mathbf{E}.
\]  

(5)

This suggests that not only the SO coupling term but also the Darwin term are considered relevant for spin-dependent properties in the weakly relativistic limit. A well-known phenomenon related to this polarization is the fine structure of the atomic spectra [16]. The first term in Eq. (3) originates from the Darwin term, representing the correction to the conventional electrical polarization due to an inhomogeneous charge distribution, whereas the second term in Eq. (3) is due to the SO coupling. The electrical polarization \( \mathbf{P} \) may be understood in terms of the Zitterbewegung of the electron [16]. It has been suggested recently that such a relativistic oscillatory motion of the electron may be measurable in a system with Rashba SO coupling [21].

The spin current and total angular momentum currents do not appear in the Maxwell equations explicitly. The SO coupling term exerts a force on the moving electron as well as a torque acting on its spin. We show now that the electrical polarization is associated with the antisymmetric part of the spin current. This property provides us a way to detect the spin current directly. To find the force and the torque on the moving electron, we start from the relativistic conservation laws of the momentum and the total angular momentum current tensors.

A rigorous NRA for the momentum current starting from the relativistic continuity equation leads to \( \dot{\mathbf{J}}^M + \nabla \cdot \mathbf{J}_0^M = \mathbf{F} \). Here the momentum and current densities are given as \( \rho^M = (1/2)(\psi^\dagger \mathbf{P} \psi - \mathbf{V} \cdot \mathbf{P}) \) and \( \mathbf{J}^M = \mathbf{J}_0^M + (\hbar/4m)\nabla \times (\psi^\dagger \sigma \mathbf{B} \psi) \) with \( \mathbf{J}_0^M = (1/4m) \times \psi^\dagger \mathbf{P} \mathbf{P} \psi + (\hbar^2/8mc^2)\{[e/h]\mathbf{E} \times (\psi^\dagger \sigma \mathbf{P} \psi - \mathbf{P} \mathbf{E})\}, \)

respectively. Here

\[
P_D = -\frac{\hbar^2}{8mc^2}[\nabla (\psi^\dagger \mathbf{P} \psi) + (1/h)\psi^\dagger \sigma \times \mathbf{B} \psi]\]

is due to the electrical polarization of carriers in the presence of the SO coupling. It is seen that the momentum current density depends on the electric field explicitly and is affected by both the SO and Darwin corrections. The total force on the electron,

\[
\mathbf{F} = \rho_0^C \mathbf{E} + \frac{1}{c} \mathbf{J}_0^C \times \mathbf{B} + (\nabla \mathbf{B}) \cdot \mathbf{m} + (\nabla \mathbf{E}) \cdot \mathbf{P},
\]  

(6)

is found to be governed by the Lorentz, “Stern-Gerlach” and an additional force due to the electrical polarization induced by the SO coupling [22]. The last term is new, which describes the force due to the electric dipole in a nonuniform electric field.

Similarly, the angular momentum density \( \rho^A \) is governed by the NRA continuity equation \( \dot{\mathbf{J}}^A + \nabla \cdot \mathbf{J}_0^A = \mathbf{T} \). Apart from a contribution of surface term, it contains two parts: (a) the orbit angular momentum density associated with the spatial rotational degree of freedom in the carriers’ orbital motion; and (b) the carriers’ intrinsic spin degree of freedom, i.e.,

\[
\rho^A = \rho^O + \rho^S + \frac{\hbar}{16mc^2} \nabla \times (\psi^\dagger \mathbf{P} \psi).
\]  

(7)

To the leading order in \( v^2/c^2 \), \( \rho^O = (1/2)(\psi^\dagger \mathbf{x} \times \mathbf{P} \psi) \), where \( \mathbf{x} = \mathbf{x} - \mathbf{x}_0 \). Here we introduce a constant vector \( \mathbf{x}_0 \) to guarantee angular moment current independent of the coordinate origin. One can easily verify the introduction of \( \mathbf{x}_0 \) does not affect the conservation of angular momentum current. The first term in Eq. (7) is therefore the orbital angular momentum density. The second term is the intrinsic spin angular momentum density \( \rho^S = (\hbar/2)\psi^\dagger \sigma \mathbf{P} \psi + \mathbf{\Sigma} \), where the first term is the carrier’s intrinsic magnetic moment density, whereas

\[
\mathbf{\Sigma} = -\frac{\hbar}{32mc^2} [\psi^\dagger C \mathbf{P} \mathbf{P} \psi + 4(\psi^\dagger \mathbf{P} \psi) \cdot \sigma \mathbf{B} \psi] + \text{h.c.}
\]

is the contribution from the carrier’s spin coupling to its orbital motion. \( \mathbf{\Sigma} \) is the order in \( v^2/c^2 \) compared with the spin density. The total angular momentum current \( \mathbf{J}^A \)}
contains both the orbital and the spin current tensors:

\[ \mathcal{J}^A = \mathbf{x} \times \mathbf{j}_0^M + \frac{\hbar}{4m} \psi^\dagger \mathbf{\Pi} \sigma \psi - \frac{\hbar}{4m} \mathbf{\nabla} \times (\psi^\dagger \mathbf{\sigma} \mathbf{\Pi} \psi \times \mathbf{x}), \]

where \( \mathbf{x} \times \mathbf{j}_0^M \) is the orbital angular momentum current. The second term is the spin current \( \mathcal{J}^S = (\hbar/4m) \psi^\dagger \mathbf{\Pi} \sigma \psi \). Both magnetic and electric fields contribute to the torque:

\[ \mathbf{T} = \mathbf{x} \times \mathbf{F} + \mathbf{m} \times \mathbf{B} + \mathbf{P} \times \mathbf{E}, \quad (8) \]

felt by the moving electron through its magnetic moment and electrical polarization. By Eq. (3), the total torque can be written as \( \mathbf{T} = \mathbf{T}^O + \mathbf{T}^S \) with

\[ \mathbf{T}^O = \mathbf{x} \times \mathbf{F} - \frac{\hbar e}{8m^2c^2} \psi^\dagger (\sigma \times \mathbf{E}) \times \mathbf{\Pi} \psi \]

and

\[ \mathbf{T}^S = \mathbf{m} \times \mathbf{B} - \frac{\hbar^2}{8m^2c^2} \left[ \mathbf{\nabla} \rho_0^S \times \mathbf{E} + \frac{e}{\hbar} \psi^\dagger (\mathbf{E} \times \mathbf{\Pi}) \times \sigma \psi \right]. \]

Here \( \mathbf{T}^O \) and \( \mathbf{T}^S \) are torques on the orbital and spin angular momentum degrees of freedom, respectively. Unlike the magnetic moment, which causes a torque only on the spin angular momentum, the electrical polarization in the electric field exerts torques on the orbit and spin angular momenta simultaneously. The second term in \( \mathbf{T}^O \) is \( \mathbf{x} \times \mathbf{F} \) and is finite for the electronic orbital motion in the presence of SO coupling. The motion of the electron therefore induces a torque on its orbital angular momentum. An electron with magnetic moment \( \mathbf{m} \) and velocity \( \mathbf{v} \) in an electric field \( \mathbf{E} \) feels a magnetic field \( \mathbf{B}' = -(\mathbf{v}/c) \times \mathbf{E} \) and torque \( \mathbf{m} \times \mathbf{B}' \) in its local coordinate frame. Up to the order of \( 1/m^2 \) the second term in \( \mathbf{T}^S \) represents \( \mathbf{m} \times \mathbf{B}' = -(\hbar e/8m^2c^2) \psi^\dagger (\mathbf{E} \times \mathbf{\Pi}) \times \sigma \psi \), where the factor 1/2 takes the Thomas precession into account.

Based on the above considerations the spin continuity equation reads

\[ \dot{\rho}^S = -\mathbf{\nabla} \cdot \mathbf{\mathcal{J}}^S + \mathbf{T}^S. \quad (9) \]

This definition differs from the symmetrized product of the Pauli matrix and the velocity operator. The term \( (\hbar/8mc) \mathbf{E} \times \psi^\dagger (\sigma_\alpha, \mathbf{m}) \mathbf{\Pi} \psi \) in \( (\hbar/4) \psi^\dagger (\sigma_\alpha, \mathbf{v}) \psi \) can be rewritten as \( (\hbar^2 e/8m^2c^2) \mathbf{E}_\alpha \psi^\dagger \psi \) with a tensor \( \mathbf{E}_\alpha = \epsilon_{\alpha \beta \gamma} \epsilon_{\beta \gamma} \mathbf{E} \). Its divergence contributes a torque in the sense of \( \mathbf{P} \times \mathbf{E} \) that reflects the inhomogeneous carrier density.

The SO coupling affects spin transport through the electrical polarization vector only, and the spin current does not appear in the Maxwell equations explicitly. However, the antisymmetric part of spin current does play an important role through the electrical polarization \( \mathbf{P} \). There is a relation between the vector

\[ \mathbf{A}^S = -\frac{\hbar}{4m} \psi^\dagger (\sigma \times \mathbf{\Pi}) \psi \]

and the antisymmetric part of the spin current tensor, i.e.,

\[ \Lambda_{\alpha \beta}^S = \epsilon_{\alpha \beta \gamma} j_{\beta \gamma}^S, \quad \text{viz.} \]

\[ \mathbf{P} = -\frac{\hbar^2}{8m^2c^2} \mathbf{\nabla} \rho_0^S + \frac{e}{2mc^2} \mathbf{A}^S. \quad (11) \]

In studying the dielectric function of a Rashba system, Rashba has found that there are some relations between spin current and polarization [23]. The relationship between spin current and torque connected to the dipole was discussed by Culcer et al. in a semiclassical transport theory [24]. However, from Eq. (11), we can demonstrate that only the antisymmetry part of the spin current \( \Lambda^S \) contributes to polarization \( \mathbf{P} \), which suggests opportunities to measure the spin current through its antisymmetry part \( \Lambda^S \).

Up to the order \( v^2/c^2 \), the force \( \mathbf{F} \) and the torque \( \mathbf{T} \) given in Eqs. (6) and (8) can be written in a compact form

\[ \left( \begin{array}{c} \mathbf{F} \\ \mathbf{T} \end{array} \right) = \left( \begin{array}{cc} F_0 & e \\ T_0 & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ -2B \end{array} \right) \times \left( \begin{array}{c} \rho_0^S \\ \Lambda^S \end{array} \right) + \frac{e}{2mc^2} \left( \begin{array}{c} 2c \mathbf{\nabla} \mathbf{B} \\ \mathbf{\nabla} \mathbf{E} \end{array} \right) \left( \begin{array}{c} \rho_0^S \\ \Lambda^S \end{array} \right), \quad (12) \]

where \( F_0 \) and \( T_0 \) are the force and the torque in the absence of the spin. An inhomogeneous electric field produces a force on the electron via \( \mathbf{P} \) that contributes to the torque on both orbit and spin angular momenta. Theoretically, a mesoscopic Stern-Gerlach spin filter device by the nonuniform SO interaction has been studied recently [25].

The general formulas can be applied to specific systems. For example, they can be used to study the properties of a two-dimensional electron gas (2DEG) with the Rashba SO coupling [26]. In this case, \( \mathbf{E} = (0, 0, E_z) \) and \( \Lambda = 0 \). The Rashba Hamiltonian is then as follows:

\[ H_R = \frac{1}{2m} p^2 \frac{\hbar}{c} (p_x \sigma_y - p_y \sigma_x). \quad (13) \]

with a parameter \( \lambda = (\hbar^2 e/4m^2c^2)E_z \), which can be enhanced by orders of magnitude compared to its vacuum value in the presence of the lattice potential for a real 2DEG, and can be tuned by the gate voltage [27]. The spin current, \( j_{\alpha \beta}^S = -i(\hbar^2/4m) \psi^\dagger \sigma_\alpha \mathbf{\Pi} \psi \), satisfy the equation \( \dot{\rho}^S + \mathbf{\nabla} \cdot \mathbf{T}^S = \mathbf{T}^S \), where the torques are \( T_{\alpha \beta}^S = -(2m\lambda/\hbar^2) j_{\alpha \beta}^S \) and \( T_{\alpha \gamma}^S = (2m\lambda/\hbar^2) (j_{\alpha \gamma}^S + j_{\gamma \alpha}^S) \). Here the Latin subscripts refer to the 2D plane, whereas Greek subscripts refer to 3D space. The Darwin term is dropped if we consider only the homogeneous 2DEG. The antisymmetric part of the in-plane components gives \( \Lambda_{\alpha \beta}^S = \delta_{\alpha \gamma}^S - \delta_{\beta \gamma}^S \), which is directly related to the electrical polarization \( P_z = (e/2mc^2) \Lambda_{\alpha \beta}^S \). Using the eigenstates of \( H_R \), we obtain

\[ P_z = \frac{\lambda}{2\pi e} \frac{\hbar^2 k_{F,0}^2}{2m} \left[ 1 - \frac{2}{3} \left( \frac{m\lambda}{\hbar^2 k_{F,0}^2} \right)^2 \right], \quad (14) \]

where \( k_{F,0} \) is the Fermi wave vector in the absence of the
SO coupling. The intrinsic electrical polarization vector is normal to the plane and depends on the strength of the SO coupling.

When a weak in-plane electric field \( E_x \) is applied to the system, the corresponding torques, induced by the electrical polarization, on the electronic spin and orbit angular momenta are given as \( T_{\alpha \beta}^S = - (e^2 / 2mc^2) \epsilon_{\alpha \beta \gamma} \epsilon_{\eta \eta \eta} E_x \delta_{\eta \gamma}^S \) and \( T_{\alpha \beta}^O = + (e^2 / 2mc^2) E_x \delta_{\eta \gamma}^O \). To the term of linear response to \( E_x \), it is found that \( T_{\gamma \alpha}^S = (1 / 2) T_{\gamma \gamma} \), \( T_{\gamma \alpha}^O = (1 / 2) E_x P_z \) by using the eigenstates of \( H_R \). The formulas of torques we used above are based on the continuity equation of angular moment current and contain the full contribution from the external fields. The torques we present here are due to the interaction of external field \( E_x \) with polarization \( P_z \), which cannot be obtained with only Rashba Hamiltonian in Eq. (13). Under these torques both the orbit and the spin angular momentum currents of 2DEG are changed.

We note that there is an essential difference between the force \( \mathbf{f} \) derived from the continuity equation of the momentum current and the transverse force on spin carriers; let us call it \( \mathbf{f}_{\perp} \). In general, a net spin current from unpolarized electrons is generated by (i) the spin-polarized electron moving under the force \( \mathbf{f} \), or (ii) the electronic orbit motion and the spin precession under the force \( \mathbf{f} \), and the torque \( \mathbf{T} \) is the covariant force associated with the electron trajectories. It follows from the equation of motion \( m \ddot{\mathbf{v}} = - m \nabla \cdot \mathbf{J}^S + \mathbf{f}_{\perp} \), where \( \mathbf{v} \) is the velocity of a wave packet of spin-polarized electrons. In a Rashba 2DEG \( v_j = (1 / 2m) \psi^\dagger \mathbf{p} \psi - (2\lambda / h^2) \epsilon_{ij} \delta_{jk} \psi^S \). The corresponding velocity tensor is given by \( \mathbf{J}^S = (1 / m) \mathbf{\Omega}^S + (2\lambda / h^2) \epsilon_{ij} \delta_{jk} \psi^S \) in which the second term can be understood as a drift velocity under the electric field. In general, the covariant force on a spin

\[
\mathbf{f}_{\perp} = (4m^2 \lambda^2 / h^2) \epsilon_{ij} \delta_{jk} \psi^S 
\]  

(15)
is obtained. This force is spin dependent. Accordingly, the opposite spins experience transverse forces in opposite directions [28–30].

We summarize the main results as follows. The conservation laws in relativistic quantum mechanics have been formulated systematically and rigorously based on Noether’s theorem. In the weakly relativistic limit the source terms in the Maxwell equations, i.e., the charge and current densities, contain contributions due to the SO coupling. The spin current has been consistently redefined with the force and the torque begot by external electromagnetic field on carriers. The relation between the electrical polarization and the antisymmetric components of the spin current is derived, which provides the probability to detect the spin current through the electrical polarization. The force and the torque on a moving spin through the electrical polarization has been identified. We demonstrated the general formulas to a Rashba 2DEG. It is shown that a torque on the spin arises and the intrinsic electrical polarization depends on the strength of the SO coupling.

The authors thank S. C. Zhang for valuable discussions and G. E. W. Bauer and C. Zhang for the critical reading and discussing of the work. This work is supported by RFDP and NNSFC Grants No. 10274069, No. 10474002, and No. 90303014.

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